*n*th Root

A number c is said to be an ***n*th root** of a if $c^n = a$.

The symbol $\sqrt{a}$ denotes the nonnegative square root of a, and the symbol $\sqrt[3]{a}$ denotes the real-number cube root of a. The symbol $\sqrt[n]{a}$ denotes the nth root of a, that is, a number whose nth power is a. The symbol $\sqrt[n]{}$ is called a **radical**, and the expression under the radical is called the **radicand**. The number n (which is omitted when it is 2) is called the **index**. Examples of roots for $n = 3, 4$, and 2, respectively, are

$$\sqrt[3]{125}, \qquad \sqrt[4]{16}, \quad \text{and} \quad \sqrt{3600}.$$

Any real number has only one real-number odd root. Any positive number has two square roots, one positive and one negative. Similarly, for any even index, a positive number has two real-number roots. The positive root is called the **principal root**. When an expression such as $\sqrt{4}$ or $\sqrt[6]{23}$ is used, it is understood to represent the principal (nonnegative) root. To denote a negative root, we use $-\sqrt{4}$, $-\sqrt[6]{23}$, and so on.

Simplify.
Assume that variables can represent any real number.

1. $\sqrt{(-11)^2}$

3. $\sqrt{16y^2}$

5. $\sqrt{(b+1)^2}$

7. $\sqrt[3]{-27x^3}$

9. $\sqrt[4]{81x^8}$

11. $\sqrt[5]{32}$

13. $\sqrt{180}$

15. $\sqrt{72}$

17. $\sqrt[3]{54}$

19. $\sqrt{128c^2d^4}$

21. $\sqrt[4]{48x^6y^4}$

23. $\sqrt{x^2 - 4x + 4}$

Simplifying Radical Expressions

Consider the expression $\sqrt{(-3)^2}$. This is equivalent to $\sqrt{9}$, or 3. Similarly, $\sqrt{3^2} = \sqrt{9} = 3$. This illustrates the first of several properties of radicals, listed below.

Properties of Radicals

Let a and b be any real numbers or expressions for which the given roots exist. For any natural numbers m and n $(n \neq 1)$:

1. If n is even, $\sqrt[n]{a^n} = |a|$.
2. If n is odd, $\sqrt[n]{a^n} = a$.
3. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$.
4. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad (b \neq 0)$.
5. $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$.

In many situations, radicands are never formed by raising negative quantities to even powers. In such cases, absolute-value notation is not required. For this reason, **we will henceforth assume that no radicands are formed by raising negative quantities to even powers**. For example, we will write $\sqrt{x^2} = x$ and $\sqrt[4]{a^5 b} = a\sqrt[4]{ab}$.

Radical expressions with the same index and the same radicand can be combined (added or subtracted) in much the same way that we combine like terms.

Simplify.
Assume that no radicands were formed by raising negative quantities to even powers.

29. $\sqrt{2x^3y}\,\sqrt{12xy}$

31. $\sqrt[3]{3x^2y}\,\sqrt[3]{36x}$

33. $\sqrt[3]{2(x+4)}\,\sqrt[3]{4(x+4)^4}$

35. $\sqrt[6]{\dfrac{m^{12}n^{24}}{64}}$

37. $\dfrac{\sqrt[3]{40m}}{\sqrt[3]{5m}}$

39. $\dfrac{\sqrt[3]{3x^2}}{\sqrt[3]{24x^5}}$

41. $\sqrt[3]{\dfrac{64a^4}{27b^3}}$

43. $\sqrt{\dfrac{7x^3}{36y^6}}$

45. $9\sqrt{50}+6\sqrt{2}$

51. $\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)$

55. $\left(1+\sqrt{3}\right)^2$

Rationalizing Denominators or Numerators

There are times when we need to remove the radicals in a denominator or a numerator. This is called **rationalizing the denominator** or **rationalizing the numerator**. It is done by multiplying by 1 in such a way as to obtain a perfect nth power.

Rationalize the denominator.

65. $\sqrt{\dfrac{2}{3}}$

67. $\dfrac{\sqrt[3]{5}}{\sqrt[3]{4}}$

71. $\dfrac{6}{3+\sqrt{5}}$

75. $\dfrac{6}{\sqrt{m}-\sqrt{n}}$

Rationalize the numerator.

77. $\dfrac{\sqrt{12}}{5}$

79. $\sqrt[3]{\dfrac{7}{2}}$

81. $\dfrac{\sqrt{11}}{\sqrt{3}}$

83. $\dfrac{9-\sqrt{5}}{3-\sqrt{3}}$

85. $\dfrac{\sqrt{a}+\sqrt{b}}{3a}$

Rational Exponents

For any real number a and any natural numbers m and n, $n \geq 2$, for which $\sqrt[n]{a}$ exists,

$$a^{1/n} = \sqrt[n]{a},$$
$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m, \quad \text{and}$$
$$a^{-m/n} = \frac{1}{a^{m/n}}.$$

We can use the definition of rational exponents to convert between radical and exponential notation.

Convert to radical notation and simplify.

87. $x^{3/4}$

89. $16^{3/4}$

91. $125^{-1/3}$

93. $a^{5/4}b^{-3/4}$

95. $m^{5/3}n^{7/3}$

Convert to exponential notation.

97. $\left(\sqrt[4]{13}\right)^5$

99. $\sqrt[3]{20^2}$

101. $\sqrt[3]{\sqrt{11}}$

103. $\sqrt{5}\,\sqrt[3]{5}$

105. $\sqrt[5]{32^2}$

Synthesis

Simplify.

127. $\sqrt{1 + x^2} + \dfrac{1}{\sqrt{1 + x^2}}$

128. $\sqrt{1 - x^2} - \dfrac{x^2}{2\sqrt{1 - x^2}}$

129. $\left(\sqrt{a^{\sqrt{a}}}\right)^{\sqrt{a}}$

130. $(2a^3b^{5/4}c^{1/7})^4 \div (54a^{-2}b^{2/3}c^{6/5})^{-1/3}$

Exercise Set R.6

1. 11 **3.** $4|y|$ **5.** $|b+1|$ **7.** $-3x$ **9.** $3x^2$ **11.** 2
13. $6\sqrt{5}$ **15.** $6\sqrt{2}$ **17.** $3\sqrt[3]{2}$ **19.** $8\sqrt{2}|c|d^2$
21. $2|x||y|\sqrt[4]{3x^2}$ **23.** $|x-2|$ **25.** $10\sqrt{3}$ **27.** $6\sqrt{11}$
29. $2x^2y\sqrt{6}$ **31.** $3x\sqrt[3]{4y}$ **33.** $2(x+4)\sqrt[3]{(x+4)^2}$
35. $\frac{m^2n^4}{2}$ **37.** 2 **39.** $\frac{1}{2x}$ **41.** $\frac{4a\sqrt[3]{a}}{3b}$ **43.** $\frac{x\sqrt{7x}}{6y^3}$
45. $51\sqrt{2}$ **47.** $4\sqrt{5}$ **49.** $-2x\sqrt{2}-12\sqrt{5x}$ **51.** 1
53. $-9-5\sqrt{15}$ **55.** $4+2\sqrt{3}$ **57.** $11-2\sqrt{30}$
59. About 13,709.5 ft **61.** **(a)** $h=\frac{a}{2}\sqrt{3}$; **(b)** $A=\frac{a^2}{4}\sqrt{3}$
63. 8 **65.** $\frac{\sqrt{6}}{3}$ **67.** $\frac{\sqrt[3]{10}}{2}$ **69.** $\frac{2\sqrt[3]{6}}{3}$ **71.** $\frac{9-3\sqrt{5}}{2}$
73. $-\frac{\sqrt{6}}{6}$ **75.** $\frac{6\sqrt{m}+6\sqrt{n}}{m-n}$ **77.** $\frac{6}{5\sqrt{3}}$ **79.** $\frac{7}{\sqrt[3]{98}}$
81. $\frac{11}{\sqrt{33}}$ **83.** $\frac{76}{27+3\sqrt{5}-9\sqrt{3}-\sqrt{15}}$

85. $\frac{a-b}{3a\sqrt{a}-3a\sqrt{b}}$ **87.** $\sqrt[4]{x^3}$ **89.** 8 **91.** $\frac{1}{5}$
93. $\frac{a\sqrt[4]{a}}{\sqrt[4]{b^3}}$, or $a\sqrt[4]{\frac{a}{b^3}}$ **95.** $mn^2\sqrt[3]{m^2n}$ **97.** $13^{5/4}$ **99.** $20^{2/3}$
101. $11^{1/6}$ **103.** $5^{5/6}$ **105.** 4 **107.** $8a^2$ **109.** $\frac{x^3}{3b^{-2}}$, or $\frac{x^3b^2}{3}$ **111.** $x\sqrt[3]{y}$ **113.** $n\sqrt[3]{mn^2}$ **115.** $a\sqrt[12]{a^5}+a^2\sqrt[12]{a}$
117. $\sqrt[6]{288}$ **119.** $\sqrt[12]{x^{11}y^7}$ **121.** $a\sqrt[6]{a^5}$
123. $(a+x)\sqrt[12]{(a+x)^{11}}$ **125.** Discussion and Writing
127. $\frac{(2+x^2)\sqrt{1+x^2}}{1+x^2}$ **129.** $a^{a/2}$