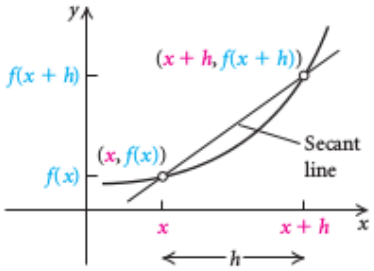
Algebra, Chapter 1.6
The Algebra of Functions

In a nutshell:

Sums, Differences, Products, and Quotients of Functions

If f and g are functions and x is in the domain of each function, then

$$(f+g)(x) = f(x) + g(x),$$
$$(f-g)(x) = f(x) - g(x),$$
$$(fg)(x) = f(x) \cdot g(x),$$
$$(f/g)(x) = f(x)/g(x), \text{ provided } g(x) \neq 0.$$



Difference Quotients

In Section 1.3, we learned that the slope of a line can be considered as an average *rate of change*. Here let's consider a nonlinear function f and draw a line through two points $(x, f(x))$ and $(x + h, f(x + h))$ as shown at left.

The slope of the line, called a **secant line**, is

$$\frac{f(x+h) - f(x)}{x + h - x},$$

which simplifies to

$$\frac{f(x+h) - f(x)}{h}.$$ Difference quotient

This ratio is called the **difference quotient**, or the **average rate of change**. In calculus, it is important to be able to find and simplify difference quotients.

Composition of Functions

The **composite function** $f \circ g$, the **composition** of f and g, is defined as

$$(f \circ g)(x) = f(g(x)),$$

where x is in the domain of g and $g(x)$ is in the domain of f.

Given that $f(x) = x^2 - 3$ and $g(x) = 2x + 1$, find each of the following, if it exists.

1. $(f + g)(5)$
2. $(fg)(0)$
3. $(f - g)(-1)$
4. $(fg)(2)$
5. $(f/g)\left(-\frac{1}{2}\right)$
6. $(f - g)(0)$
7. $(fg)\left(-\frac{1}{2}\right)$
8. $(f/g)\left(-\sqrt{3}\right)$
9. $(g - f)(-1)$
10. $(g/f)\left(-\frac{1}{2}\right)$

Given that $h(x) = x + 4$ and $g(x) = \sqrt{x - 1}$, find each of the following, if it exists.

11. $(h - g)(-4)$
12. $(gh)(10)$
13. $(g/h)(1)$
14. $(h/g)(1)$
15. $(g + h)(1)$
16. $(hg)(3)$

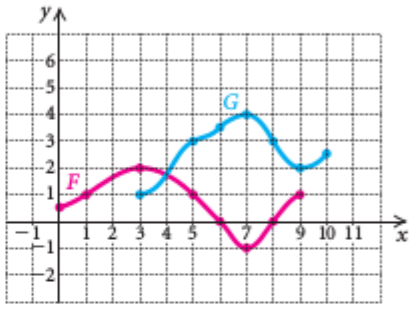
For each pair of functions in Exercises 17–32:

a) *Find the domain of f, g, $f + g$, $f - g$, fg, ff, f/g, and g/f.*

b) *Find $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, $(ff)(x)$, $(f/g)(x)$, and $(g/f)(x)$.*

17. $f(x) = 2x + 3,\ g(x) = 3 - 5x$
18. $f(x) = -x + 1,\ g(x) = 4x - 2$
19. $f(x) = x - 3,\ g(x) = \sqrt{x + 4}$
20. $f(x) = x + 2,\ g(x) = \sqrt{x - 1}$
21. $f(x) = 2x - 1,\ g(x) = -2x^2$
22. $f(x) = x^2 - 1,\ g(x) = 2x + 5$
23. $f(x) = \sqrt{x - 3},\quad g(x) = \sqrt{x + 3}$
24. $f(x) = \sqrt{x},\ g(x) = \sqrt{2 - x}$
25. $f(x) = x + 1,\ g(x) = |x|$
26. $f(x) = 4|x|,\ g(x) = 1 - x$
27. $f(x) = x^3,\ g(x) = 2x^2 + 5x - 3$
28. $f(x) = x^2 - 4,\ g(x) = x^3$
29. $f(x) = \dfrac{4}{x + 1},\ g(x) = \dfrac{1}{6 - x}$
30. $f(x) = 2x^2,\ g(x) = \dfrac{2}{x - 5}$
31. $f(x) = \dfrac{1}{x},\ g(x) = x - 3$
32. $f(x) = \sqrt{x + 6},\ g(x) = \dfrac{1}{x}$

In Exercises 33–38, consider the functions F and G as shown in the following graph.



33. Find the domain of F, the domain of G, and the domain of $F + G$.
34. Find the domain of $F - G$, FG, and F/G.
35. Find the domain of G/F.
36. Graph $F + G$.
37. Graph $G - F$.
38. Graph $F - G$.

39. *Total Cost, Revenue, and Profit.* In economics, functions that involve revenue, cost, and profit are used. For example, suppose that $R(x)$ and $C(x)$ denote the total revenue and the total cost, respectively, of producing a new kind of tool for King Hardware Wholesalers. Then the difference

$$P(x) = R(x) - C(x)$$

represents the total profit for producing x tools. Given

$$R(x) = 60x - 0.4x^2 \quad \text{and} \quad C(x) = 3x + 13,$$

find each of the following.

a) $P(x)$

b) $R(100)$, $C(100)$, and $P(100)$

c) Using a graphing calculator, graph the three functions in the viewing window $[0, 160, 0, 3000]$.

40. *Total Cost, Revenue, and Profit.* Given that

$$R(x) = 200x - x^2 \quad \text{and} \quad C(x) = 5000 + 8x,$$

for a new radio produced by Clear Communication, find each of the following. (See Exercise 39.)

a) $P(x)$

b) $R(175)$, $C(175)$, and $P(175)$

c) Using a graphing calculator, graph the three functions in the viewing window $[0, 200, 0, 10{,}000]$.

For each function f, construct and simplify the difference quotient

$$\frac{f(x+h) - f(x)}{h}.$$

41. $f(x) = x^2 + 1$

42. $f(x) = 2 - x^2$

43. $f(x) = 3x - 5$

44. $f(x) = -\frac{1}{2}x + 7$

45. $f(x) = 3x^2 - 2x + 1$

46. $f(x) = 5x^2 + 4x$

47. $f(x) = 4 + 5|x|$

48. $f(x) = 2|x| + 3x$

49. $f(x) = x^3$

50. $f(x) = x^3 - 2x$

51. $f(x) = \dfrac{x-4}{x+3}$

52. $f(x) = \dfrac{x}{2-x}$

Given that $f(x) = 3x + 1$, $g(x) = x^2 - 2x - 6$, and $h(x) = x^3$, find each of the following.

53. $(f \circ g)(-1)$

54. $(g \circ f)(-2)$

55. $(h \circ f)(1)$

56. $(g \circ h)\left(\frac{1}{2}\right)$

57. $(g \circ f)(5)$

58. $(f \circ g)\left(\frac{1}{3}\right)$

59. $(f \circ h)(-3)$

60. $(h \circ g)(3)$

Find $(f \circ g)(x)$ and $(g \circ f)(x)$ and the domain of each.

61. $f(x) = x + 3$, $g(x) = x - 3$

62. $f(x) = \frac{4}{5}x$, $g(x) = \frac{5}{4}x$

63. $f(x) = \dfrac{4}{1-5x}$, $g(x) = \dfrac{1}{x}$

64. $f(x) = \dfrac{6}{x}$, $g(x) = \dfrac{1}{2x+1}$

65. $f(x) = 3x - 7$, $g(x) = \dfrac{x+7}{3}$

66. $f(x) = \frac{2}{3}x - \frac{4}{5}$, $g(x) = 1.5x + 1.2$

67. $f(x) = 2x + 1$, $g(x) = \sqrt{x}$

68. $f(x) = \sqrt{x-4}$, $g(x) = \dfrac{2}{x}$

69. $f(x) = 20$, $g(x) = 0.05$

70. $f(x) = x^4$, $g(x) = \sqrt[4]{x}$

71. $f(x) = \sqrt{x+5}$, $g(x) = x^2 - 5$

72. $f(x) = x^5 - 2$, $g(x) = \sqrt[5]{x+2}$

73. $f(x) = x^2 + 2$, $g(x) = \sqrt{3-x}$

74. $f(x) = 1 - x^2$, $g(x) = \sqrt{x^2 - 25}$

75. $f(x) = \dfrac{1-x}{x}$, $g(x) = \dfrac{1}{1+x}$

76. $f(x) = \dfrac{x^2-1}{x^2+1}$, $g(x) = \dfrac{3x-4}{5x-2}$

77. $f(x) = x^3 - 5x^2 + 3x + 7$, $g(x) = x + 1$

78. $f(x) = x - 1$, $g(x) = x^3 + 2x^2 - 3x - 9$

Find $f(x)$ and $g(x)$ such that $h(x) = (f \circ g)(x)$. Answers may vary.

79. $h(x) = (4 + 3x)^5$

80. $h(x) = \sqrt[3]{x^2 - 8}$

81. $h(x) = \dfrac{1}{(x-2)^4}$

82. $h(x) = \dfrac{1}{\sqrt{3x+7}}$

83. $h(x) = \dfrac{x^3-1}{x^3+1}$

84. $h(x) = |9x^2 - 4|$

85. $h(x) = \left(\dfrac{2+x^3}{2-x^3}\right)^6$

86. $h(x) = \left(\sqrt{x} - 3\right)^4$

87. $h(x) = \sqrt{\dfrac{x-5}{x+2}}$

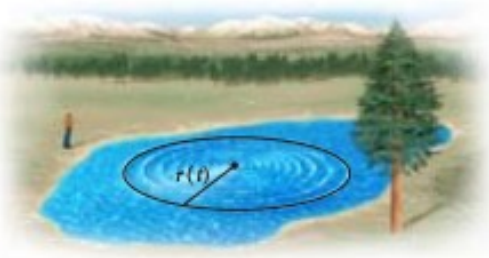
88. $h(x) = \sqrt{1 + \sqrt{1+x}}$

89. $h(x) = (x+2)^3 - 5(x+2)^2 + 3(x+2) - 1$

90. $h(x) = 2(x-1)^{5/3} + 5(x-1)^{2/3}$

91. *Dress Sizes.* A dress that is size x in France is size $s(x)$ in the United States, where $s(x) = x - 32$. A dress that is size x in the United States is size $y(x)$ in Italy, where $y(x) = 2(x + 12)$. Find a function that will convert French dress sizes to Italian dress sizes.

92. *Ripple Spread.* A stone is thrown into a pond, creating a circular ripple that spreads over the pond in such a way that the radius is increasing at the rate of 3 ft/sec.



a) Find a function $r(t)$ for the radius in terms of t.
b) Find a function $A(r)$ for the area of the ripple in terms of the radius r.
c) Find $(A \circ r)(t)$. Explain the meaning of this function.

Collaborative Discussion and Writing

93. If $g(x) = b$, where b is a positive constant, describe how the graphs of $y = h(x)$ and $y = (h - g)(x)$ will differ.

94. Explain which values of x must be excluded from the domain of $(f \circ g)(x)$ and the domain of $(g \circ f)(x)$.

Skill Maintenance

Consider the following linear equations. Without graphing them, answer the questions below.

a) $y = x$
b) $y = -5x + 4$
c) $y = \frac{2}{3}x + 1$
d) $y = -0.1x + 6$
e) $y = 3x - 5$
f) $y = -x - 1$
g) $2x - 3y = 6$
h) $6x + 3y = 9$

95. Which, if any, have y-intercept $(0, 1)$?

96. Which, if any, have the same y-intercept?

97. Which slope down from left to right?

98. Which has the steepest slope?

99. Which pass(es) through the origin?

100. Which, if any, have the same slope?

101. Which, if any, are parallel?

102. Which, if any, are perpendicular?

Synthesis

103. Write equations of two functions f and g such that $f \circ g = g \circ f = x$. (In Section 4.1, we will study inverse functions. If $f \circ g = g \circ f = x$, functions f and g are *inverses* of each other.)

104. Write equations for two functions f and g such that the domain of $f - g$ is

$$\{x \mid x \neq -7 \text{ and } x \neq 3\}.$$

105. Find the domain of $(h/g)(x)$ given that

$$h(x) = \frac{5x}{3x - 7} \quad \text{and} \quad g(x) = \frac{x^4 - 1}{5x - 15}.$$

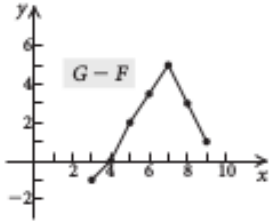
106. For functions h and f, find the domain of $h + f$, $h - f$, hf, and h/f if

$$h = \left\{(-4, 13), (-1, 7), (0, 5), \left(\tfrac{5}{2}, 0\right), (3, -5)\right\}, \quad \text{and}$$

$$f = \{(-4, -7), (-2, -5), (0, -3), (3, 0), (5, 2), (9, 6)\}.$$

Exercise Set 1.6

1. 33 **3.** -1 **5.** Does not exist **7.** 0 **9.** 1
11. Does not exist **13.** 0 **15.** 5
17. (a) Domain of f, g, $f+g$, $f-g$, fg, and ff: $(-\infty,\infty)$;
domain of f/g: $\left(-\infty,\frac{3}{5}\right)\cup\left(\frac{3}{5},\infty\right)$;
domain of g/f: $\left(-\infty,-\frac{3}{2}\right)\cup\left(-\frac{3}{2},\infty\right)$;
(b) $(f+g)(x)=-3x+6$; $(f-g)(x)=7x$;
$(fg)(x)=-10x^2-9x+9$; $(ff)(x)=4x^2+12x+9$;
$(f/g)(x)=\dfrac{2x+3}{3-5x}$; $(g/f)(x)=\dfrac{3-5x}{2x+3}$
19. (a) Domain of f: $(-\infty,\infty)$; domain of g: $[-4,\infty)$;
domain of $f+g$, $f-g$, and fg: $[-4,\infty)$;
domain of ff: $(-\infty,\infty)$; domain of f/g: $(-4,\infty)$;
domain of g/f: $[-4,3)\cup(3,\infty)$;
(b) $(f+g)(x)=x-3+\sqrt{x+4}$;
$(f-g)(x)=x-3-\sqrt{x+4}$; $(fg)(x)=(x-3)\sqrt{x+4}$;
$(ff)(x)=x^2-6x+9$; $(f/g)(x)=\dfrac{x-3}{\sqrt{x+4}}$;
$(g/f)(x)=\dfrac{\sqrt{x+4}}{x-3}$
21. (a) Domain of f, g, $f+g$, $f-g$, fg, and ff: $(-\infty,\infty)$;
domain of f/g: $(-\infty,0)\cup(0,\infty)$;
domain of g/f: $\left(-\infty,\frac{1}{2}\right)\cup\left(\frac{1}{2},\infty\right)$
(b) $(f+g)(x)=-2x^2+2x-1$;
$(f-g)(x)=2x^2+2x-1$; $(fg)(x)=-4x^3+2x^2$;
$(ff)(x)=4x^2-4x+1$; $(f/g)(x)=\dfrac{2x-1}{-2x^2}$;
$(g/f)(x)=\dfrac{-2x^2}{2x-1}$
23. (a) Domain of f: $[3,\infty)$; domain of g: $[-3,\infty)$;
domain of $f+g$, $f-g$, fg, and ff: $[3,\infty)$;
domain of f/g: $[3,\infty)$; domain of g/f: $(3,\infty)$;
(b) $(f+g)(x)=\sqrt{x-3}+\sqrt{x+3}$;
$(f-g)(x)=\sqrt{x-3}-\sqrt{x+3}$; $(fg)(x)=\sqrt{x^2-9}$;
$(ff)(x)=|x-3|$; $(f/g)(x)=\dfrac{\sqrt{x-3}}{\sqrt{x+3}}$; $(g/f)(x)=\dfrac{\sqrt{x+3}}{\sqrt{x-3}}$
25. (a) Domain of f, g, $f+g$, $f-g$, fg, and ff: $(-\infty,\infty)$;
domain of f/g: $(-\infty,0)\cup(0,\infty)$;
domain of g/f: $(-\infty,-1)\cup(-1,\infty)$;
(b) $(f+g)(x)=x+1+|x|$; $(f-g)(x)=x+1-|x|$;
$(fg)(x)=(x+1)|x|$; $(ff)(x)=x^2+2x+1$;
$(f/g)(x)=\dfrac{x+1}{|x|}$; $(g/f)(x)=\dfrac{|x|}{x+1}$
27. (a) Domain of f, g, $f+g$, $f-g$, fg, and ff: $(-\infty,\infty)$;
domain of f/g: $(-\infty,-3)\cup\left(-3,\frac{1}{2}\right)\cup\left(\frac{1}{2},\infty\right)$;
domain of g/f: $(-\infty,0)\cup(0,\infty)$;
(b) $(f+g)(x)=x^3+2x^2+5x-3$;
$(f-g)(x)=x^3-2x^2-5x+3$;
$(fg)(x)=2x^5+5x^4-3x^3$; $(ff)(x)=x^6$;
$(f/g)(x)=\dfrac{x^3}{2x^2+5x-3}$; $(g/f)(x)=\dfrac{2x^2+5x-3}{x^3}$
29. (a) Domain of f: $(-\infty,-1)\cup(-1,\infty)$;
domain of g: $(-\infty,6)\cup(6,\infty)$; domain of $f+g$, $f-g$, and fg: $(-\infty,-1)\cup(-1,6)\cup(6,\infty)$;
domain of ff: $(-\infty,-1)\cup(-1,\infty)$;
domain of f/g and g/f: $(-\infty,-1)\cup(-1,6)\cup(6,\infty)$;
(b) $(f+g)(x)=\dfrac{4}{x+1}+\dfrac{1}{6-x}$;
$(f-g)(x)=\dfrac{4}{x+1}-\dfrac{1}{6-x}$; $(fg)(x)=\dfrac{4}{(x+1)(6-x)}$;
$(ff)(x)=\dfrac{16}{(x+1)^2}$; $(f/g)(x)=\dfrac{4(6-x)}{x+1}$;
$(g/f)(x)=\dfrac{x+1}{4(6-x)}$
31. (a) Domain of f: $(-\infty,0)\cup(0,\infty)$;
domain of g: $(-\infty,\infty)$; domain of $f+g$, $f-g$, fg, and ff: $(-\infty,0)\cup(0,\infty)$; domain of f/g: $(-\infty,0)\cup(0,3)\cup(3,\infty)$;
domain of g/f: $(-\infty,0)\cup(0,\infty)$;
(b) $(f+g)(x)=\dfrac{1}{x}+x-3$; $(f-g)(x)=\dfrac{1}{x}-x+3$;
$(fg)(x)=1-\dfrac{3}{x}$; $(ff)(x)=\dfrac{1}{x^2}$; $(f/g)(x)=\dfrac{1}{x(x-3)}$;
$(g/f)(x)=x(x-3)$
33. Domain of F: $[0,9]$; domain of G: $[3,10]$; domain of $F+G$: $[3,9]$ **35.** $[3,6)\cup(6,8)\cup(8,9]$
37.



39. (a) $P(x)=-0.4x^2+57x-13$; (b) $R(100)=2000$;
$C(100)=313$; $P(100)=1687$; (c) Left to the student
41. $2x+h$ **43.** 3 **45.** $6x+3h-2$
47. $\dfrac{5|x+h|-5|x|}{h}$ **49.** $3x^2+3xh+h^2$
51. $\dfrac{7}{(x+h+3)(x+3)}$ **53.** -8 **55.** 64 **57.** 218
59. -80 **61.** $(f\circ g)(x)=(g\circ f)(x)=x$;
domain of $f\circ g$ and $g\circ f$: $(-\infty,\infty)$
63. $(f\circ g)(x)=\dfrac{4x}{x-5}$; $(g\circ f)(x)=\dfrac{1-5x}{4}$;
domain of $f\circ g$: $(-\infty,0)\cup(0,5)\cup(5,\infty)$;
domain of $g\circ f$: $\left(-\infty,\frac{1}{5}\right)\cup\left(\frac{1}{5},\infty\right)$
65. $(f\circ g)(x)=(g\circ f)(x)=x$;
domain of $f\circ g$ and $g\circ f$: $(-\infty,\infty)$
67. $(f\circ g)(x)=2\sqrt{x}+1$; $(g\circ f)(x)=\sqrt{2x+1}$;
domain of $f\circ g$: $[0,\infty)$; domain of $g\circ f$: $\left[-\frac{1}{2},\infty\right)$
69. $(f\circ g)(x)=20$; $(g\circ f)(x)=0.05$; domain of $f\circ g$ and $g\circ f$: $(-\infty,\infty)$