(d) $u \cdot v = (7-2i)(\overline{1+i}) + (2+5i)(\overline{-3-6i})$
$= (7-2i)(1-i) + (2+5i)(-3+6i) = 5 - 9i - 36 - 3i = -31 - 12i$

(e) $\|u\| = \sqrt{7^2 + (-2)^2 + 2^2 + 5^2} = \sqrt{82}$ and $\|v\| = \sqrt{1^2 + 1^2 + (-3)^2 + (-6)^2} = \sqrt{47}$

1.40. Prove: For any vectors $u, v \in \mathbf{C}^n$ and any scalar $z \in \mathbf{C}$, (i) $u \cdot v = \overline{v \cdot u}$, (ii) $(zu) \cdot v = z(u \cdot v)$, (iii) $u \cdot (zv) = \bar{z}(u \cdot v)$.

Suppose $u = (z_1, z_2, \ldots, z_n)$ and $v = (w_1, w_2, \ldots, w_n)$.

(i) Using the properties of the conjugate,

$$\overline{v \cdot u} = \overline{w_1\bar{z}_1 + w_2\bar{z}_2 + \cdots + w_n\bar{z}_n} = \overline{w_1\bar{z}_1} + \overline{w_2\bar{z}_2} + \cdots + \overline{w_n\bar{z}_n}$$
$$= \bar{w}_1 z_1 + \bar{w}_2 z_2 + \cdots + \bar{w}_n z_n = z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n = u \cdot v$$

(ii) Because $zu = (zz_1, zz_2, \ldots, zz_n)$,

$$(zu) \cdot v = zz_1\bar{w}_1 + zz_2\bar{w}_2 + \cdots + zz_n\bar{w}_n = z(z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n) = z(u \cdot v)$$

(Compare with Theorem 1.2 on vectors in $\mathbf{R}^n$.)

(iii) Using (i) and (ii),

$$u \cdot (zv) = \overline{(zv) \cdot u} = \overline{z(v \cdot u)} = \bar{z}(\overline{v \cdot u}) = \bar{z}(u \cdot v)$$

SUPPLEMENTARY PROBLEMS

Vectors in $\mathbf{R}^n$

1.41. Let $u = (1, -2, 4)$, $v = (3, 5, 1)$, $w = (2, 1, -3)$. Find:

(a) $3u - 2v$; (b) $5u + 3v - 4w$; (c) $u \cdot v$, $u \cdot w$, $v \cdot w$; (d) $\|u\|$, $\|v\|$;
(e) $\cos\theta$, where θ is the angle between u and v; (f) $d(u, v)$; (g) $\text{proj}(u, v)$.

1.42. Repeat Problem 1.41 for vectors $u = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$, $w = \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}$.

1.43. Let $u = (2, -5, 4, 6, -3)$ and $v = (5, -2, 1, -7, -4)$. Find:

(a) $4u - 3v$; (b) $5u + 2v$; (c) $u \cdot v$; (d) $\|u\|$ and $\|v\|$; (e) $\text{proj}(u, v)$; (f) $d(u, v)$.

1.44. Normalize each vector:

(a) $u = (5, -7)$; (b) $v = (1, 2, -2, 4)$; (c) $w = \left(\frac{1}{2}, -\frac{1}{3}, \frac{3}{4}\right)$.

1.45. Let $u = (1, 2, -2)$, $v = (3, -12, 4)$, and $k = -3$.

(a) Find $\|u\|$, $\|v\|$, $\|u + v\|$, $\|ku\|$.
(b) Verify that $\|ku\| = |k|\|u\|$ and $\|u + v\| \le \|u\| + \|v\|$.

1.46. Find x and y where:

(a) $(x,\ y + 1) = (y - 2,\ 6)$; (b) $x(2, y) = y(1, -2)$.

1.47. Find x, y, z where $(x,\ y + 1,\ y + z) = (2x + y,\ 4,\ 3z)$.

1.48. Write $v = (2, 5)$ as a linear combination of u_1 and u_2, where:

(a) $u_1 = (1, 2)$ and $u_2 = (3, 5)$;
(b) $u_1 = (3, -4)$ and $u_2 = (2, -3)$.

1.49. Write $v = \begin{bmatrix} 9 \\ -3 \\ 16 \end{bmatrix}$ as a linear combination of $u_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$.

1.50. Find k so that u and v are orthogonal, where:

(a) $u = (3, k, -2)$, $v = (6, -4, -3)$;
(b) $u = (5, k, -4, 2)$, $v = (1, -3, 2, 2k)$;
(c) $u = (1,\ 7,\ k+2,\ -2)$, $v = (3, k, -3, k)$.

Located Vectors, Hyperplanes, Lines in $\mathbf{R}^n$

1.51. Find the vector v identified with the directed line segment $\overrightarrow{PQ}$ for the points:

(a) $P(2, 3, -7)$ and $Q(1, -6, -5)$ in $\mathbf{R}^3$;
(b) $P(1, -8, -4, 6)$ and $Q(3, -5, 2, -4)$ in $\mathbf{R}^4$.

1.52. Find an equation of the hyperplane H in $\mathbf{R}^4$ that:

(a) contains $P(1, 2, -3, 2)$ and is normal to $u = [2, 3, -5, 6]$;
(b) contains $P(3, -1, 2, 5)$ and is parallel to $2x_1 - 3x_2 + 5x_3 - 7x_4 = 4$.

1.53. Find a parametric representation of the line in $\mathbf{R}^4$ that:

(a) passes through the points $P(1, 2, 1, 2)$ and $Q(3, -5, 7, -9)$;
(b) passes through $P(1, 1, 3, 3)$ and is perpendicular to the hyperplane $2x_1 + 4x_2 + 6x_3 - 8x_4 = 5$.

Spatial Vectors (Vectors in $\mathbf{R}^3$), ijk Notation

1.54. Given $u = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $v = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, $w = 4\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$. Find:

(a) $2u - 3v$; (b) $3u + 4v - 2w$; (c) $u \cdot v$, $u \cdot w$, $v \cdot w$; (d) $\|u\|$, $\|v\|$, $\|w\|$.

1.55. Find the equation of the plane H:

(a) with normal $\mathbf{N} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ and containing the point $P(1, 2, -3)$;
(b) parallel to $4x + 3y - 2z = 11$ and containing the point $Q(2, -1, 3)$.

1.56. Find the (parametric) equation of the line L:

(a) through the point $P(2, 5, -3)$ and in the direction of $v = 4\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$;
(b) perpendicular to the plane $2x - 3y + 7z = 4$ and containing $P(1, -5, 7)$.

1.57. Consider the following curve C in $\mathbf{R}^3$ where $0 \le t \le 5$:

$$F(t) = t^3\mathbf{i} - t^2\mathbf{j} + (2t - 3)\mathbf{k}$$

(a) Find the point P on C corresponding to $t = 2$.
(b) Find the initial point Q and the terminal point Q'.
(c) Find the unit tangent vector $\mathbf{T}$ to the curve C when $t = 2$.

1.58. Consider a moving body B whose position at time t is given by $R(t) = t^2\mathbf{i} + t^3\mathbf{j} + 3t\mathbf{k}$. [Then $V(t) = dR(t)/dt$ and $A(t) = dV(t)/dt$ denote, respectively, the velocity and acceleration of B.] When $t = 1$, find for the body B:

(a) position; (b) velocity v; (c) speed s; (d) acceleration a.

1.59. Find a normal vector **N** and the tangent plane H to each surface at the given point:

(a) surface $x^2y + 3yz = 20$ and point $P(1, 3, 2)$;
(b) surface $x^2 + 3y^2 - 5z^2 = 160$ and point $P(3, -2, 1)$.

Cross Product

1.60. Evaluate the following determinants and negative of determinants of order two:

(a) $\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix}$, $\begin{vmatrix} 3 & -6 \\ 1 & -4 \end{vmatrix}$, $\begin{vmatrix} -4 & -2 \\ 7 & -3 \end{vmatrix}$

(b) $-\begin{vmatrix} 6 & 4 \\ 7 & 5 \end{vmatrix}$, $-\begin{vmatrix} 1 & -3 \\ 2 & 4 \end{vmatrix}$, $-\begin{vmatrix} 8 & -3 \\ -6 & -2 \end{vmatrix}$

1.61. Given $u = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $v = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, $w = 4\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$, find:

(a) $u \times v$, (b) $u \times w$, (c) $v \times w$.

1.62. Given $u = [2, 1, 3]$, $v = [4, -2, 2]$, $w = [1, 1, 5]$, find:

(a) $u \times v$, (b) $u \times w$, (c) $v \times w$.

1.63. Find the volume V of the parallelopiped formed by the vectors u, v, w appearing in:

(a) Problem 1.60 (b) Problem 1.61.

1.64. Find a unit vector u orthogonal to:

(a) $v = [1, 2, 3]$ and $w = [1, -1, 2]$;
(b) $v = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $w = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

1.65. Prove the following properties of the cross product:

(a) $u \times v = -(v \times u)$
(b) $u \times u = 0$ for any vector u
(c) $(ku) \times v = k(u \times v) = u \times (kv)$
(d) $u \times (v + w) = (u \times v) + (u \times w)$
(e) $(v + w) \times u = (v \times u) + (w \times u)$
(f) $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$

Complex Numbers

1.66. Simplify:

(a) $(4 - 7i)(9 + 2i)$; (b) $(3 - 5i)^2$; (c) $\dfrac{1}{4 - 7i}$; (d) $\dfrac{9 + 2i}{3 - 5i}$; (e) $(1 - i)^3$.

1.67. Simplify: (a) $\dfrac{1}{2i}$; (b) $\dfrac{2 + 3i}{7 - 3i}$; (c) i^{15}, i^{25}, i^{34}; (d) $\left(\dfrac{1}{3 - i}\right)^2$.

1.68. Let $z = 2 - 5i$ and $w = 7 + 3i$. Find:

(a) $v + w$; (b) zw; (c) z/w; (d) $\bar{z}, \bar{w}$; (e) $|z|$, $|w|$.

1.69. Show that for complex numbers z and w:

(a) $\operatorname{Re} z = \frac{1}{2}(z + \bar{z})$, (b) $\operatorname{Im} z = \frac{1}{2}(z - \bar{z})$, (c) $zw = 0$ implies $z = 0$ or $w = 0$.

Vectors in C^n

1.70. Let $u = (1 + 7i,\ 2 - 6i)$ and $v = (5 - 2i,\ 3 - 4i)$. Find:

(a) $u + v$ (b) $(3 + i)u$ (c) $2iu + (4 + 7i)v$ (d) $u \cdot v$ (e) $\|u\|$ and $\|v\|$.

1.71. Prove: For any vectors u, v, w in $\mathbf{C}^n$:

(a) $(u+v)\cdot w = u\cdot w + v\cdot w$, (b) $w\cdot(u+v) = w\cdot u + w\cdot v$.

1.72. Prove that the norm in $\mathbf{C}^n$ satisfies the following laws:

$[N_1]$ For any vector u, $\|u\| \geq 0$; and $\|u\| = 0$ if and only if $u = 0$.
$[N_2]$ For any vector u and complex number z, $\|zu\| = |z|\|u\|$.
$[N_3]$ For any vectors u and v, $\|u+v\| \leq \|u\| + \|v\|$.

ANSWERS TO SUPPLEMENTARY PROBLEMS

1.41. (a) $(-3,-16,4)$; (b) (6,1,35); (c) $-3, 12, 8$; (d) $\sqrt{21}, \sqrt{35}, \sqrt{14}$;
(e) $-3/\sqrt{21}\sqrt{35}$; (f) $\sqrt{62}$; (g) $-\frac{3}{35}\,(3,5,1) = (-\frac{9}{35}, -\frac{15}{35}, -\frac{3}{35})$

1.42. (Column vectors) (a) $(-1,7,-22)$; (b) $(-1,26,-29)$; (c) $-15,-27,34$;
(d) $\sqrt{26}, \sqrt{30}$; (e) $-15/(\sqrt{26}\sqrt{30})$; (f) $\sqrt{86}$; (g) $-\frac{15}{30}\,v = (-1, -\frac{1}{2}, -\frac{5}{2})$

1.43. (a) $(-13,-14,13,45,0)$; (b) $(20,-29,22,16,-23)$; (c) -6; (d) $\sqrt{90}, \sqrt{95}$;
(e) $-\frac{6}{95}\,v$; (f) $\sqrt{167}$

1.44. (a) $(5/\sqrt{76},\ 9/\sqrt{76})$; (b) $(\frac{1}{5}, \frac{2}{5}, -\frac{2}{5}, \frac{4}{5})$; (c) $(6/\sqrt{133},\ -4\sqrt{133},\ 9\sqrt{133})$

1.45. (a) $3,\ 13,\ \sqrt{120},\ 9$

1.46. (a) $x=-3,\ y=5$; (b) $x=0,\ y=0$, and $x=1,\ y=2$

1.47. $x=-3, \quad y=3, \quad z=\frac{3}{2}$

1.48. (a) $v = 5u_1 - u_2$; (b) $v = 16u_1 - 23u_2$

1.49. $v = 3u_1 - u_2 + 2u_3$

1.50. (a) 6; (b) 3; (c) $\frac{3}{2}$

1.51. (a) $v = [-1,-9,2]$; (b) $[2,3,6,-10]$

1.52. (a) $2x_1 + 3x_2 - 5x_3 + 6x_4 = 35$; (b) $2x_1 - 3x_2 + 5x_3 - 7x_4 = -16$

1.53. (a) $[2t+1,\ -7t+2,\ 6t+1,\ -11t+2]$; (b) $[2t+1,\ 4t+1,\ 6t+3,\ -8t+3]$

1.54. (a) $-23\mathbf{j} + 13\mathbf{k}$; (b) $9\mathbf{i} - 6\mathbf{j} - 10\mathbf{k}$; (c) $-20,-12,37$; (d) $\sqrt{29}, \sqrt{38}, \sqrt{69}$

1.55. (a) $3x - 4y + 5z = -20$; (b) $4x + 3y - 2z = -1$

1.56. (a) $[4t+2,\ -5t+5,\ 7t-3]$; (b) $[2t+1,\ -3t-5,\ 7t+7]$

1.57. (a) $P = F(2) = 8\mathbf{i} - 4\mathbf{j} + \mathbf{k}$; (b) $Q = F(0) = -3\mathbf{k},\ Q' = F(5) = 125\mathbf{i} - 25\mathbf{j} + 7\mathbf{k}$;
(c) $\mathbf{T} = (6\mathbf{i} - 2\mathbf{j} + \mathbf{k})/\sqrt{41}$

1.58. (a) $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$; (b) $2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$; (c) $\sqrt{17}$; (d) $2\mathbf{i} + 6\mathbf{j}$

1.59. (a) $\mathbf{N} = 6\mathbf{i} + 7\mathbf{j} + 9\mathbf{k},\ 6x + 7y + 9z = 45$; (b) $\mathbf{N} = 6\mathbf{i} - 12\mathbf{j} - 10\mathbf{k},\ 3x - 6y - 5z = 16$

1.60. (a) $-3, -6, 26$; (b) $-2, -10, 34$

1.61. (a) $2\mathbf{i} + 13\mathbf{j} + 23\mathbf{k}$; (b) $-22\mathbf{i} + 2\mathbf{j} + 37\mathbf{k}$; (c) $31\mathbf{i} - 16\mathbf{j} - 6\mathbf{k}$

1.62. (a) $[5, 8, -6]$; (b) $[2, -7, 1]$; (c) $[-7, -18, 5]$

1.63. (a) 143; (b) 17

1.64. (a) $(7, 1, -3)/\sqrt{59}$; (b) $(5\mathbf{i} + 11\mathbf{j} - 2\mathbf{k})/\sqrt{150}$

1.66. (a) $50 - 55i$; (b) $-16 - 30i$; (c) $\frac{1}{65}(4 + 7i)$; (d) $\frac{1}{2}(1 + 3i)$; (e) $-2 - 2i$

1.67. (a) $-\frac{1}{2}i$; (b) $\frac{1}{58}(5 + 27i)$; (c) $-1, i, -1$; (d) $\frac{1}{50}(4 + 3i)$

1.68. (a) $9 - 2i$; (b) $29 - 29i$; (c) $\frac{1}{61}(-1 - 41i)$; (d) $2 + 5i,\ 7 - 3i$; (e) $\sqrt{29}, \sqrt{58}$

1.69. (c) *Hint:* If $zw = 0$, then $|zw| = |z||w| = |0| = 0$

1.70. (a) $(6 + 5i,\ 5 - 10i)$; (b) $(-4 + 22i,\ 12 - 16i)$; (c) $(-8 - 41i,\ -4 - 33i)$;
(d) $12 + 2i$; (e) $\sqrt{90}, \sqrt{54}$

SUPPLEMENTARY PROBLEMS

Algebra of Matrices

Problems 2.38–2.41 refer to the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 \\ -6 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -3 & 4 \\ 2 & 6 & -5 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 7 & -1 \\ 4 & -8 & 9 \end{bmatrix}$$

2.38. Find (a) $5A - 2B$, (b) $2A + 3B$, (c) $2C - 3D$.

2.39. Find (a) AB and $(AB)C$, (b) BC and $A(BC)$. [Note that $(AB)C = A(BC)$.]

2.40. Find (a) A^2 and A^3, (b) AD and BD, (c) CD.

2.41. Find (a) A^T, (b) B^T, (c) $(AB)^T$, (d) $A^T B^T$. [Note that $A^T B^T \neq (AB)^T$.]
Problems 2.42 and 2.43 refer to the following matrices:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & -3 \\ -1 & -2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 5 & -1 & -4 & 2 \\ -1 & 0 & 0 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

2.42. Find (a) $3A - 4B$, (b) AC, (c) BC, (d) AD, (e) BD, (f) CD.

2.43. Find (a) A^T, (b) $A^T B$, (c) $A^T C$.

2.44. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. Find a 2×3 matrix B with distinct nonzero entries such that $AB = 0$.

2.45 Let $e_1 = [1, 0, 0]$, $e_2 = [0, 1, 0]$, $e_3 = [0, 0, 1]$, and $A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{bmatrix}$. Find $e_1 A$, $e_2 A$, $e_3 A$.

2.46. Let $e_i = [0, \ldots, 0, 1, 0, \ldots, 0]$, where 1 is the ith entry. Show

(a) $e_i A = A_i$, ith row of A.

(b) $Be_j^T = B^j$, jth column of B.

(c) If $e_i A = e_i B$, for each i, then $A = B$.

(d) If $Ae_j^T = Be_j^T$, for each j, then $A = B$.

2.47. Prove Theorem 2.2(iii) and (iv): (iii) $(B + C)A = BA + CA$, (iv) $k(AB) = (kA)B = A(kB)$.

2.48. Prove Theorem 2.3: (i) $(A + B)^T = A^T + B^T$, (ii) $(A^T)^T = A$, (iii) $(kA)^T = kA^T$.

2.49. Show (a) If A has a zero row, then AB has a zero row. (b) If B has a zero column, then AB has a zero column.

Square Matrices, Inverses

2.50. Find the diagonal and trace of each of the following matrices:

(a) $A = \begin{bmatrix} 2 & -5 & 8 \\ 3 & -6 & -7 \\ 4 & 0 & -1 \end{bmatrix}$, (b) $B = \begin{bmatrix} 1 & 3 & -4 \\ 6 & 1 & 7 \\ 2 & -5 & -1 \end{bmatrix}$, (c) $C = \begin{bmatrix} 4 & 3 & -6 \\ 2 & -5 & 0 \end{bmatrix}$

Problems 2.51–2.53 refer to $A = \begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -2 \\ 1 & -6 \end{bmatrix}$, $C = \begin{bmatrix} 6 & -4 \\ 3 & -2 \end{bmatrix}$.

2.51. Find (a) A^2 and A^3, (b) $f(A)$ and $g(A)$, where

$$f(x) = x^3 - 2x^2 - 5, \qquad g(x) = x^2 - 3x + 17.$$

2.52. Find (a) B^2 and B^3, (b) $f(B)$ and $g(B)$, where

$$f(x) = x^2 + 2x - 22, \qquad g(x) = x^2 - 3x - 6.$$

2.53. Find a nonzero column vector u such that $Cu = 4u$.

2.54. Find the inverse of each of the following matrices (if it exists):

$$A = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \qquad C = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}, \qquad D = \begin{bmatrix} 5 & -2 \\ 6 & -3 \end{bmatrix}$$

2.55. Find the inverses of $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 3 & -2 \end{bmatrix}$. [*Hint:* See Problem 2.19.]

2.56. Suppose A is invertible. Show that if $AB = AC$, then $B = C$. Give an example of a nonzero matrix A such that $AB = AC$ but $B \neq C$.

2.57. Find 2×2 invertible matrices A and B such that $A + B \neq 0$ and $A + B$ is not invertible.

2.58. Show (a) A is invertible if and only if A^T is invertible. (b) The operations of inversion and transpose commute; that is, $(A^T)^{-1} = (A^{-1})^T$. (c) If A has a zero row or zero column, then A is not invertible.

Diagonal and triangular matrices

2.59. Let $A = \text{diag}(1, 2, -3)$ and $B = \text{diag}(2, -5, 0)$. Find

(a) AB, A^2, B^2; (b) $f(A)$, where $f(x) = x^2 + 4x - 3$; (c) A^{-1} and B^{-1}.

2.60. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. (a) Find A^n. (b) Find B^n.

2.61. Find all real triangular matrices A such that $A^2 = B$, where (a) $B = \begin{bmatrix} 4 & 21 \\ 0 & 25 \end{bmatrix}$, (b) $B = \begin{bmatrix} 1 & 4 \\ 0 & -9 \end{bmatrix}$.

2.62. Let $A = \begin{bmatrix} 5 & 2 \\ 0 & k \end{bmatrix}$. Find all numbers k for which A is a root of the polynomial:

(a) $f(x) = x^2 - 7x + 10$, (b) $g(x) = x^2 - 25$, (c) $h(x) = x^2 - 4$.

2.63. Let $B = \begin{bmatrix} 1 & 0 \\ 26 & 27 \end{bmatrix}$. Find a matrix A such that $A^3 = B$.

2.64. Let $B = \begin{bmatrix} 1 & 8 & 5 \\ 0 & 9 & 5 \\ 0 & 0 & 4 \end{bmatrix}$. Find a triangular matrix A with positive diagonal entries such that $A^2 = B$.

2.65. Using only the elements 0 and 1, find the number of 3×3 matrices that are (a) diagonal, (b) upper triangular, (c) nonsingular and upper triangular. Generalize to $n \times n$ matrices.

2.66. Let $D_k = kI$, the scalar matrix belonging to the scalar k. Show

(a) $D_k A = kA$, (b) $BD_k = kB$, (c) $D_k + D_{k'} = D_{k+k'}$, (d) $D_k D_{k'} = D_{kk'}$

2.67. Suppose $AB = C$, where A and C are upper triangular.

(a) Find 2×2 nonzero matrices A, B, C, where B is not upper triangular.

(b) Suppose A is also invertible. Show that B must also be upper triangular.

Special Types of Real Matrices

2.68. Find x, y, z such that A is symmetric, where

(a) $A = \begin{bmatrix} 2 & x & 3 \\ 4 & 5 & y \\ z & 1 & 7 \end{bmatrix}$, (b) $A = \begin{bmatrix} 7 & -6 & 2x \\ y & z & -2 \\ x & -2 & 5 \end{bmatrix}$.

2.69. Suppose A is a square matrix. Show (a) $A + A^T$ is symmetric, (b) $A - A^T$ is skew-symmetric, (c) $A = B + C$, where B is symmetric and C is skew-symmetric.

2.70. Write $A = \begin{bmatrix} 4 & 5 \\ 1 & 3 \end{bmatrix}$ as the sum of a symmetric matrix B and a skew-symmetric matrix C.

2.71. Suppose A and B are symmetric. Show that the following are also symmetric:

(a) $A + B$; (b) kA, for any scalar k; (c) A^2;
(d) A^n, for $n > 0$; (e) $f(A)$, for any polynomial $f(x)$.

2.72. Find a 2×2 orthogonal matrix P whose first row is a multiple of

(a) $(3, -4)$, (b) $(1, 2)$.

2.73. Find a 3×3 orthogonal matrix P whose first two rows are multiples of

(a) $(1, 2, 3)$ and $(0, -2, 3)$, (b) $(1, 3, 1)$ and $(1, 0, -1)$.

2.74. Suppose A and B are orthogonal matrices. Show that A^T, A^{-1}, AB are also orthogonal.

2.75. Which of the following matrices are normal? $A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Complex Matrices

2.76. Find real numbers x, y, z such that A is Hermitian, where $A = \begin{bmatrix} 3 & x + 2i & yi \\ 3 - 2i & 0 & 1 + zi \\ yi & 1 - xi & -1 \end{bmatrix}$.

2.77. Suppose A is a complex matrix. Show that AA^H and $A^H A$ are Hermitian.

2.78. Let A be a square matrix. Show that (a) $A + A^H$ is Hermitian, (b) $A - A^H$ is skew-Hermitian, (c) $A = B + C$, where B is Hermitian and C is skew-Hermitian.

2.79. Determine which of the following matrices are unitary:

$$A = \begin{bmatrix} i/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -i/2 \end{bmatrix}, \qquad B = \frac{1}{2}\begin{bmatrix} 1 + i & 1 - i \\ 1 - i & 1 + i \end{bmatrix}, \qquad C = \frac{1}{2}\begin{bmatrix} 1 & -i & -1 + i \\ i & 1 & 1 + i \\ 1 + i & -1 + i & 0 \end{bmatrix}$$

2.80. Suppose A and B are unitary. Show that A^H, A^{-1}, AB are unitary.

2.81. Determine which of the following matrices are normal: $A = \begin{bmatrix} 3 + 4i & 1 \\ i & 2 + 3i \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 - i & i \end{bmatrix}$.

Block Matrices

2.82. Let $U = \left[\begin{array}{cc|ccc} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ \hline 0 & 0 & 5 & 1 & 2 \\ 0 & 0 & 3 & 4 & 1 \end{array}\right]$ and $V = \left[\begin{array}{cc|cc} 3 & -2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ \hline 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & -4 & 1 \end{array}\right]$.

(a) Find UV using block multiplication. (b) Are U and V block diagonal matrices?
(c) Is UV block diagonal?

2.83. Partition each of the following matrices so that it becomes a square block matrix with as many diagonal blocks as possible:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

2.84. Find M^2 and M^3 for (a) $M = \left[\begin{array}{c|cc|c} 2 & 0 & 0 & 0 \\ \hline 0 & 1 & 4 & 0 \\ 0 & 2 & 1 & 0 \\ \hline 0 & 0 & 0 & 3 \end{array}\right]$, (b) $M = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ \hline 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 5 \end{array}\right]$.

2.85. For each matrix M in Problem 2.84, find $f(M)$ where $f(x) = x^2 + 4x - 5$.

2.86. Suppose $U = [U_{ik}]$ and $V = [V_{kj}]$ are block matrices for which UV is defined and the number of columns of each block U_{ik} is equal to the number of rows of each block V_{kj}. Show that $UV = [W_{ij}]$, where $W_{ij} = \sum_k U_{ik} V_{kj}$.

2.87. Suppose M and N are block diagonal matrices where corresponding blocks have the same size, say $M = \text{diag}(A_i)$ and $N = \text{diag}(B_i)$. Show

(i) $M + N = \text{diag}(A_i + B_i)$, (iii) $MN = \text{diag}(A_i B_i)$,
(ii) $kM = \text{diag}(kA_i)$, (iv) $f(M) = \text{diag}(f(A_i))$ for any polynomial $f(x)$.

ANSWERS TO SUPPLEMENTARY PROBLEMS

Notation: $A = [R_1; \quad R_2; \quad \ldots]$ denotes a matrix A with rows $R_1, R_2, \ldots$.

2.38. (a) $[-5, 10; \quad 27, -34]$, (b) $[17, 4; \quad -12, 13]$, (c) $[-7, -27, 11; \quad -8, 36, -37]$

2.39. (a) $[-7, 14; \quad 39, -28]$, $[21, 105, -98; \quad -17, -285, 296]$
(b) $[5, -15, 20; \quad 8, 60, -59]$, $[21, 105, -98; \quad -17, -285, 296]$

2.40. (a) $[7, -6; \quad -9, 22]$, $[-11, 38; \quad 57, -106]$;
(b) $[11, -9, 17; \quad -7, 53, -39]$, $[15, 35, -5; \quad 10, -98, 69]$; (c) not defined

2.41. (a) $[1, 3; \quad 2, -4]$, (b) $[5, -6; \quad 0, 7]$, (c) $[-7, 39; \quad 14, -28]$, (d) $[5, 15; \quad 10, -40]$

2.42. (a) $[-13, -3, 18; \quad 4, 17, 0]$, (b) $[-5, -2, 4, 5; \quad 11, -3, -12, 18]$,
(c) $[11, -12, 0, -5; \quad -15, 5, 8, 4]$, (d) $[9; \quad 9]$, (e) $[-1; \quad 9]$, (f) not defined

2.43. (a) $[1, 0; \quad -1, 3; \quad 2, 4]$, (b) $[4, 0, -3; \quad -7, -6, 12; \quad 4, -8, 6]$, (c) not defined

2.44. $[2, 4, 6; \quad -1, -2, -3]$

2.45. $[a_1, a_2, a_3, a_4], \quad [b_1, b_2, b_3, b_4], \quad [c_1, c_2, c_3, c_4]$

2.50. (a) $2, -6, -1, \operatorname{tr}(A) = -5$, (b) $1, 1, -1, \operatorname{tr}(B) = 1$, (c) not defined

2.51. (a) $[-11, -15; \quad 9, -14], \quad [-67, 40; \quad -24, -59]$, (b) $[-50, 70; \quad -42, -36]$, $g(A) = 0$

2.52. (a) $[14, 4; \quad -2, 34], \quad [60, -52; \quad 26, -200]$, (b) $f(B) = 0$, $[-4, 10; \quad -5, 46]$

2.53. $u = [2a, a]^T$

2.54. $[3, -4; \quad -5, 7], \quad [-\frac{5}{2}, \frac{3}{2}; \quad 2, -1]$, not defined, $[1, -\frac{2}{3}; \quad 2, -\frac{5}{3}]$

2.55. $[1, 1, -1; \quad 2, -5, 3; \quad -1, 2, -1], \quad [1, 1, 0; \quad -1, -3, 1; \quad -1, -4, 1]$

2.56. $A = [1, 2; \quad 1, 2]$, $B = [0, 0; \quad 1, 1]$, $C = [2, 2; \quad 0, 0]$

2.57. $A = [1, 2; \quad 0, 3]$; $B = [4, 3; \quad 3, 0]$

2.58. (c) *Hint:* Use Problem 2.48

2.59. (a) $AB = \operatorname{diag}(2, -10, 0)$, $A^2 = \operatorname{diag}(1, 4, 9)$, $B^2 = \operatorname{diag}(4, 25, 0)$;
(b) $f(A) = \operatorname{diag}(2, 9, -6)$; (c) $A^{-1} = \operatorname{diag}(1, \frac{1}{2}, -\frac{1}{3})$, C^{-1} does not exist

2.60. (a) $[1, 2n; \quad 0, 1]$, (b) $[1, n, \frac{1}{2}n(n-1); \quad 0, 1, n; \quad 0, 0, 1]$

2.61. (a) $[2, 3; \quad 0, 5], \quad [-2, -3; \quad 0, -5], \quad [2, -7; \quad 0, -5], \quad [-2, 7; \quad 0, 5]$, (b) none

2.62. (a) $k = 2$, (b) $k = -5$, (c) none

2.63. $[1, 0; \quad 2, 3]$

2.64. $[1, 2, 1; \quad 0, 3, 1; \quad 0, 0, 2]$

2.65. All entries below the diagonal must be 0 to be upper triangular, and all diagonal entries must be 1 to be nonsingular.
(a) $8 \ (2^n)$, (b) $2^6 \ (2^{n(n+1)/2})$, (c) $2^3 \ (2^{n(n-1)/2})$

2.67. (a) $A = [1, 1; \quad 0, 0]$, $B = [1, 2; \quad 3, 4]$, $C = [4, 6; \quad 0, 0]$

2.68. (a) $x = 4$, $y = 1$, $z = 3$; (b) $x = 0$, $y = -6$, z any real number

2.69. (c) *Hint:* Let $B = \frac{1}{2}(A + A^T)$ and $C = \frac{1}{2}(A - A^T)$.

2.70. $B = [4, 3; \quad 3, 3]$, $C = [0, 2; \quad -2, 0]$

2.72. (a) $[\frac{3}{5}, -\frac{4}{5}; \quad \frac{4}{5}, \frac{3}{5}]$, (b) $[1/\sqrt{5}, 2/\sqrt{5}; \quad 2/\sqrt{5}, -1/\sqrt{5}]$

2.73. (a) $[1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14}; \quad 0, -2/\sqrt{13}, 3/\sqrt{13}; \quad 12/\sqrt{157}, -3/\sqrt{157}, -2/\sqrt{157}]$
(b) $[1/\sqrt{11}, 3/\sqrt{11}, 1/\sqrt{11}; \quad 1/\sqrt{2}, 0, -1/\sqrt{2}; \quad 3/\sqrt{22}, -2/\sqrt{22}, 3/\sqrt{22}]$

2.75. A, C

2.76. $x = 3$, $y = 0$, $z = 3$

2.78. (c) *Hint:* Let $B = \frac{1}{2}(A + A^H)$ and $C = \frac{1}{2}(A - A^H)$.

2.79. A, B, C

2.81. A

2.82. (a) $UV = \text{diag}([7, 6; \quad 17, 10]; [-1, 9; \quad 7, -5])$; (b) no; (c) yes

2.83. A: line between first and second rows (columns);
B: line between second and third rows (columns) and between fourth and fifth rows (columns);
C: C itself—no further partitioning of C is possible.

2.84. (a) $M^2 = \text{diag}([4], [9, 8; \quad 4, 9], \quad [9])$,
$M^3 = \text{diag}([8], [25, 44; \quad 22, 25], [27])$
(b) $M^2 = \text{diag}([3, 4; \quad 8, 11], \quad [9, 12; \quad 24, 33])$
$M^3 = \text{diag}([11, 15; \quad 30, 41], \quad [57, 78; \quad 156, 213])$

2.85. (a) $\text{diag}([7], \quad [8, 24; \quad 12, 8], [16])$, (b) $\text{diag}([2, 8; \quad 16, 181], \quad [8, 20; \quad 40, 48])$